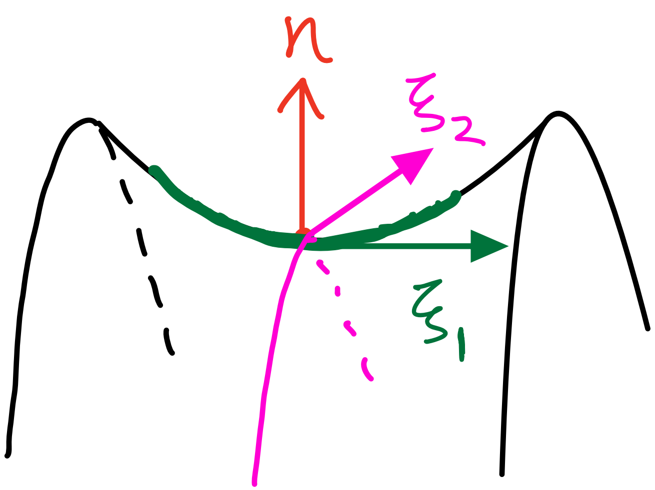
Definition 3.4.6 (Principal curvatures and principal directions). *Let S be a regular surface and $p \in S$. Let $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in T_pS$ be two linearly independent eigenvectors of the differential $d\mathbf{n}_p : T_pS \to T_pS$ of Gauss map at p and κ_1, κ_2 be negative of the associated eigenvalues respectively. In other words,*

$$\begin{cases} d\mathbf{n}_p(\boldsymbol{\xi}_1) = -\kappa_1\boldsymbol{\xi}_1 \\ d\mathbf{n}_p(\boldsymbol{\xi}_2) = -\kappa_2\boldsymbol{\xi}_2 \end{cases}.$$

Then we say that κ_1, κ_2 are the **principal curvatures** *of S at p, and $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2$ are the corresponding* **principal directions**.

Theorem 3.4.8. *Let S be a regular surface in $\mathbb{R}^3$ and $p \in S$. Then there exists principal directions $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in T_pS$ which constitute an orthonormal basis for T_pS.*



$$dn_p(\xi_1) = \overbrace{-\kappa_1}^{<0}\xi_1 \Rightarrow \kappa_1 > 0$$

$$dn_p(\xi_2) = \underbrace{-\kappa_2}_{>0}\xi_2 \Rightarrow \kappa_2 < 0$$

$\kappa > 0$ if the curve bends towards $\vec{n}$

Theorem 3.4.10. *Let S be a regular surface and K be the Gaussian curvature of S. Then for any $p \in S$,*

$$K(p) = \det(d\mathbf{n}_p) = \kappa_1\kappa_2 = \frac{\det(\mathrm{II})}{\det(\mathrm{I})}$$

Definition 3.4.11 (Mean curvature)**.** *Let S be a regular surface and $d\mathbf{n}_p$ be the differential of Gauss map at $p \in S$. The* **mean curvature** *of S at p is*

$$H = -\frac{1}{2}\mathrm{tr}(d\mathbf{n}_p) = \frac{1}{2}(\kappa_1+\kappa_2) = \frac{1}{2}\mathrm{tr}((II)(I^{-1})) = \frac{1}{2}\left(\frac{gE-2fF+eG}{EG-F^2}\right).$$

tr = trace = sum of diagonal entries

$\det(dn_p)$ and $\mathrm{tr}(dn_p)$ can be computed using $-(\mathrm{II})\mathrm{I}^{-1}$ or $\begin{bmatrix} -\kappa_1 & 0 \\ 0 & -\kappa_2 \end{bmatrix}$

$$n : S \longrightarrow S^2$$

$$dn_p : T_pS \longrightarrow T_{n(p)}S^2 = T_pS$$

Matrix representation of dn_p with respect to X_u, X_v

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = -(\mathrm{II})\mathrm{I}^{-1}$$

$$= -\frac{1}{EG-F^2}\begin{pmatrix} eG-fF & fE-eF \\ fG-gF & gE-fF \end{pmatrix}$$

$$n_u = dn_p(X_u) = aX_u + bX_v$$

$$n_v = dn_p(X_v) = cX_u + dX_v$$

Matrix representation of dn_p with respect to ξ_1, ξ_2

$$\begin{bmatrix} -\kappa_1 & 0 \\ 0 & -\kappa_2 \end{bmatrix}$$

$$dn_p(\xi_1) = -\kappa_1\xi_1 + 0\xi_2$$

$$dn_p(\xi_2) = 0\xi_1 - \kappa_2\xi_2$$

<u>Rmk</u>

- $-\kappa_1, -\kappa_2$ are eigenvalues of $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = -(\mathrm{II})\mathrm{I}^{-1}$

  roots of $\det\begin{bmatrix} t-a & -b \\ -c & t-d \end{bmatrix}$

- $\mathrm{tr}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a+d$

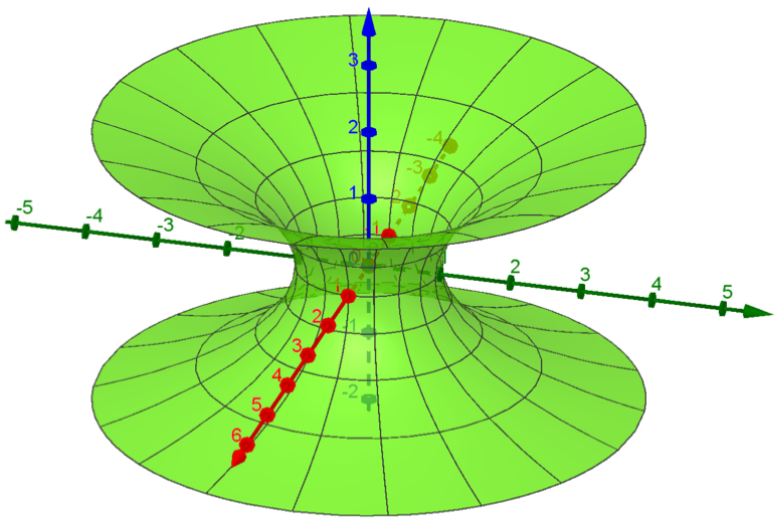
Definition 3.4.12 (Minimal surface). *Let S be a regular surface in $\mathbb{R}^3$ and H be the mean curvature of S. We say that S is a* **minimal surface** *if $H = 0$ at every point of S.*

Theorem 3.4.13. *Let S be a minimal surface with parametrization $\mathbf{x} : D \to \mathbb{R}^3$ such that $\mathbf{x}$ can be extended continuously to the boundary. Then S has the minimum surface area among all surfaces with the same boundary of S.*

Example 3.4.14. *Show that the catenoid parametrized by*

$$\mathbf{x}(\theta, v) = (\cosh v \cos\theta, \cosh v \sin\theta, v), \;\; 1 < \theta < 2\pi, v \in \mathbb{R},$$

is a minimal surface.



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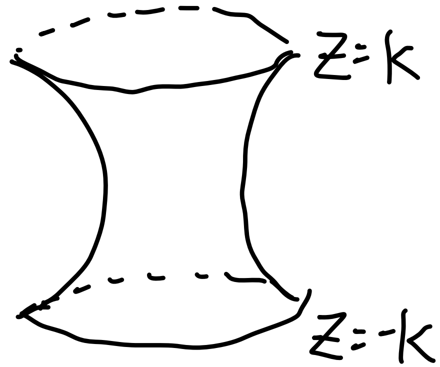
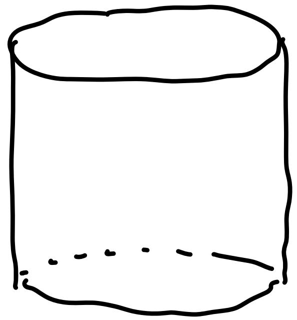


$A = \pi(2k + \sinh 2k)$

Proof. We have

$$\begin{aligned}
\mathbf{x}_\theta &= (-\cosh v \sin\theta, \cosh v \cos\theta, 0)\\
\mathbf{x}_v &= (\sinh v \cos\theta, \sinh v \sin\theta, 1)\\
\mathbf{x}_\theta \times \mathbf{x}_v &= (\cosh v \cos\theta, \cosh v \sin\theta, -\cosh v \sinh v)\\
\|\mathbf{x}_\theta \times \mathbf{x}_v\|^2 &= \cosh^2 v + \cosh^2 v \sinh^2 v = \cosh^2 v(1 + \sinh^2 v) = \cosh^4 v\\
\mathbf{n} &= (\mathrm{sech} v \cos\theta, \mathrm{sech} v \sin\theta, \tanh v)\\
\mathbf{x}_{\theta\theta} &= (-\cosh v \cos\theta, -\cosh v \sin\theta, 0)\\
\mathbf{x}_{\theta v} &= (-\sinh v \sin\theta, \sinh v \cos\theta, 0)\\
\mathbf{x}_{vv} &= (\cosh v \cos\theta, \cosh v \sin\theta, 0).
\end{aligned}$$

$E = \langle x_\theta, x_\theta \rangle$



$A = 4\pi k \cosh k$

Then the first and second fundamental forms are

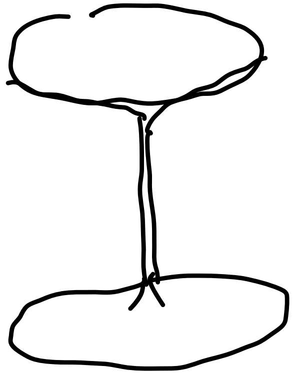
$e = \langle x_{\theta\theta}, n \rangle$

$$\begin{aligned}
I &= \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} \cosh^2 v & 0 \\ 0 & \cosh^2 v \end{pmatrix}\\
II &= \begin{pmatrix} e & f \\ f & g \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}
\end{aligned}$$

Thus the mean curvature is

$$H = \frac{1}{2}\left(\frac{gE - 2fF + eG}{EG - F^2}\right) = \frac{1}{2}\left(\frac{\cosh^2 v - \cosh^2 v}{\cosh^4 v}\right) = 0.$$

Therefore the catenoid is a minimal surface. ∎



$A \to 2\pi \cosh^2 k$

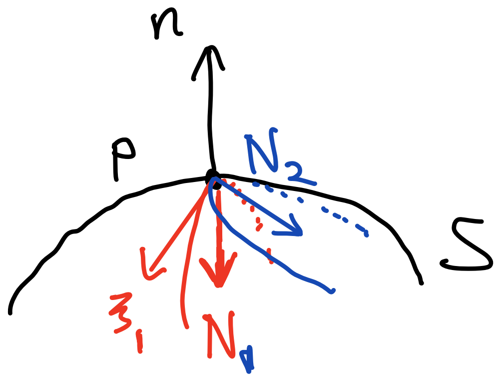
Theorem 3.4.15. *Let S be a regular surface and $p \in S$ be a point on S. Let C be a regular parametrized curve passing through p. Then we have*

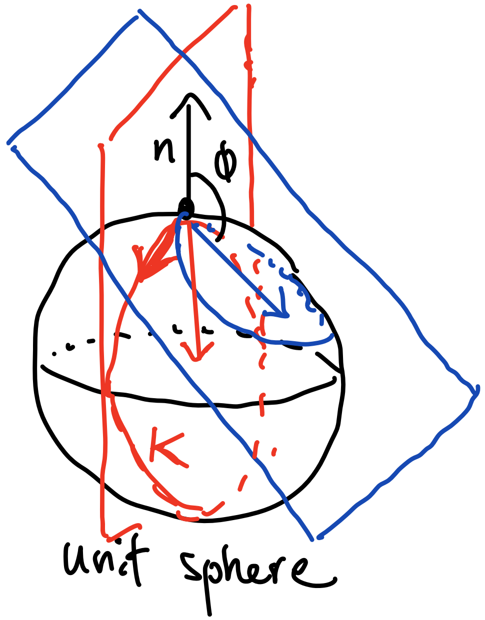
$$\kappa \cos\phi = -\langle \mathbf{T}, d\mathbf{n}_p(\mathbf{T})\rangle$$

where $\mathbf{T}, \kappa$ are the unit tangent vector, signed curvature of C at p respectively, $d\mathbf{n}_p$ is the differential of Gauss map of S at p and ϕ is the angle between the unit normal vector $\mathbf{N}$ of C and the unit normal vector $\mathbf{n}$ of S at p. Furthermore if $\mathbf{T} = \alpha\mathbf{x}_u + \beta\mathbf{x}_v \in T_pS$, then we have

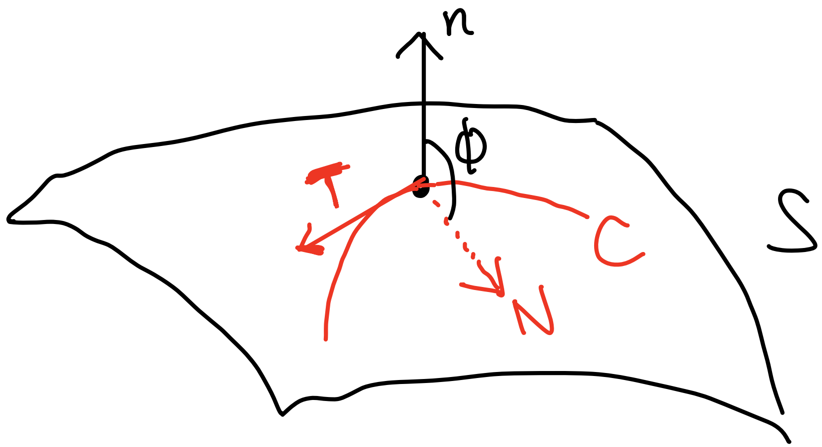
$$\kappa\cos\phi = (\alpha \quad \beta)\, II \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

where II is the second fundamental form.





$\kappa < \kappa$



κ = signed curvature of C

$$\kappa \cos\phi = -\langle T, dn_p(T)\rangle$$

<u>Def</u> normal curvature at p

$$\kappa_n(v) = -\langle v, dn_p(v)\rangle$$

Definition 3.4.16 (Normal curvature)**.** *Let S be a regular surface and p be a point on S. Let $\mathbf{v} \in T_pS$ be a unit vector tangent to the surface S at p. The* **normal curvature** *of S at p along $\mathbf{v}$ is*

$$\kappa_n(\mathbf{v}) = \kappa \cos\phi = -\langle \mathbf{v}, d\mathbf{n}_p(\mathbf{v})\rangle$$

where κ is the curvature of a curve C which passes through p and has unit tangent vector equals to $\mathbf{v}$, and ϕ is the angle between the unit normal vectors $\mathbf{N}$ and $\mathbf{n}$ of C and S at p respectively.

Theorem 3.4.17. *Let S be a regular surface and $p \in S$ be a point on S. Let $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2$ be the principal directions which constitute an orthonormal basis for T_pS and κ_1, κ_2 be the associated principal curvatures at p respectively. Let $\mathbf{v} \in T_pS$ be a unit vector tangent to S at p with $\mathbf{v} = \cos\theta\boldsymbol{\xi}_1 + \sin\theta\boldsymbol{\xi}_2$ where θ is the angle between $\mathbf{v}$ and $\boldsymbol{\xi}_1$. Then the normal curvature of S at p along $\mathbf{v}$ is*

$$\kappa_n(\mathbf{v}) = \kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta.$$

Theorem 3.4.18. *Let S be a regular surface and $p \in S$. Let $\kappa_1 \leq \kappa_2$ be the principal curvatures of S at p which associate with two orthogonal principal directions. Then for any unit vector $\mathbf{v} \in T_pS$ tangent to S at p, the normal curvature $\kappa_n(\mathbf{v})$ along $\mathbf{v}$ satisfies*

$$\kappa_1 \leq \kappa_n(\mathbf{v}) \leq \kappa_2.$$

Theorem 3.4.19. *Let S be a regular surface parametrized by $\mathbf{x}(u, v)$ and K be the Gaussian curvature of S.*

1.
$$K = \frac{\det(II)}{\det(I)}$$
where I and II are the first fundamental forms of S.

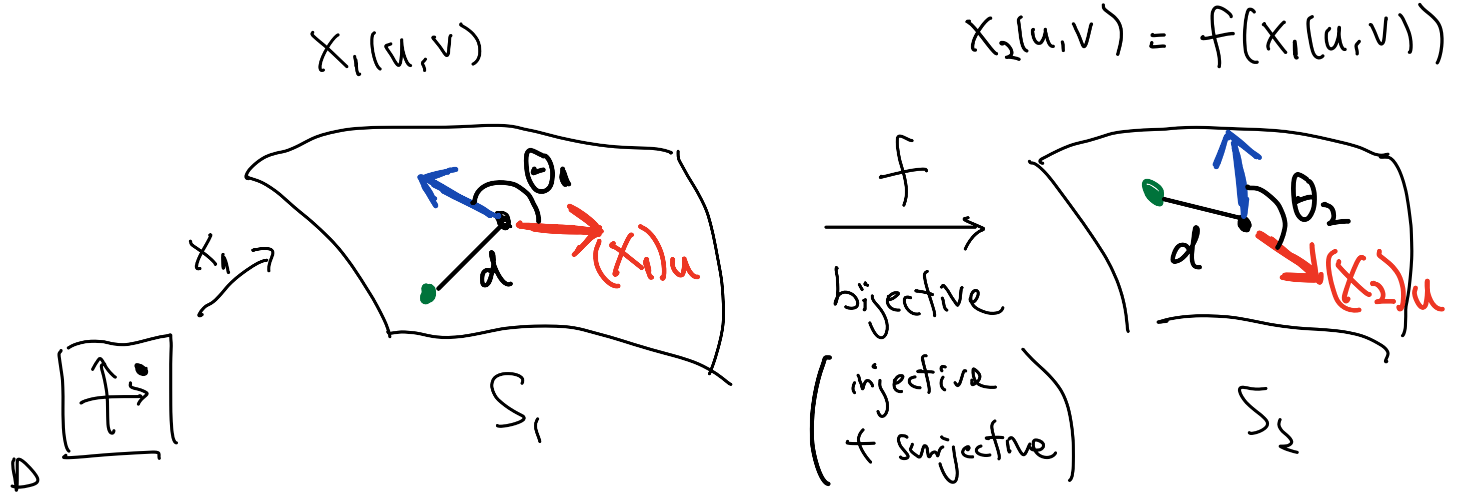
2.
$$\mathbf{n}_u \times \mathbf{n}_v = K\mathbf{x}_u \times \mathbf{x}_v$$
where $\mathbf{n}$ is the unit normal vector of S.

3.
$$K = \frac{d\sigma}{dA}$$
where A and σ are the signed area function on S and S^2 respectively.

4.
$$K = \kappa_1 \kappa_2$$
where κ_1, κ_2 are the principal curvatures associated with two orthogonal principal directions.

3.5 Theorema egregium (remarkable theorem)

Let S_1 be a regular surface and $f : S_1 \to S_2$ be a differentiable bijective map from S_1 to another regular surface S_2. Then any regular parametrization $\mathbf{x}_1(u,v)$ of S_1 induces a parametrization of S_2 by $\mathbf{x}_2(u,v) = f \circ \mathbf{x}_1(u,v) = f(\mathbf{x}_1(u,v))$. Furthermore the first fundamental forms $I_1(u,v)$ and $I_2(u,v)$ on S_1 and S_2 with respect to $\mathbf{x}_1(u,v)$ and $\mathbf{x}_2(u,v)$ can both be considered as matrix valued functions of u, v. We say that $f : S_1 \to S_2$ is an isometry if $I_1(u,v) = I_2(u,v)$ for any u, v.



$$I = \begin{bmatrix} \langle X_u, X_u \rangle & \langle X_v, X_u \rangle \\ \langle X_u, X_v \rangle & \langle X_v, X_v \rangle \end{bmatrix}$$

If $I_1(u,v) = I_2(u,v)$ for all $(u,v) \in D$, f is called isometry $\Rightarrow$ $\|(X_1)_u\| = \|(X_2)_u\|$, $\|(X_1)_v\| = \|(X_2)_v\|$, $\theta_1 = \theta_2$

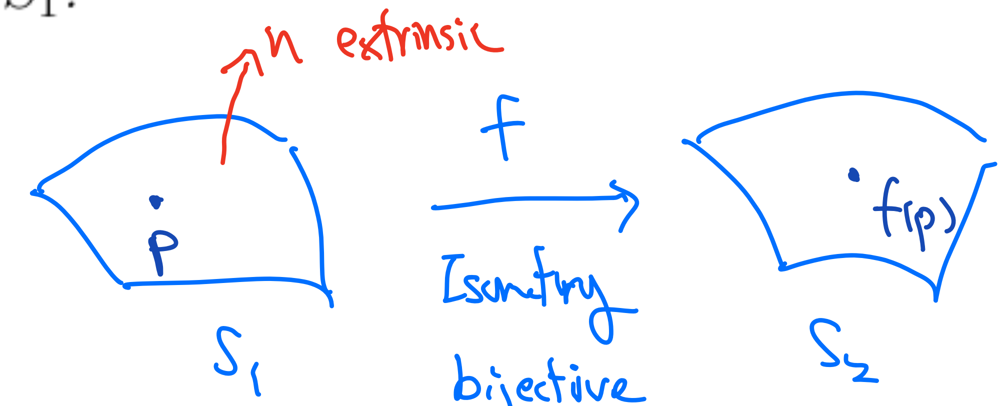
S_1, S_2 are called isometric

Definition 3.5.1 (Isometry)**.** *Let* S_1 *and* S_2 *be regular surfaces. Let* $f : S_1 \to S_2$ *be a differentiable bijective map from* S_1 *to* S_2*. We say that a map* $f : S_1 \to S_2$ *is an* **isometry** *if* $I_1(u, v) = I_2(u, v)$ *for any* u, v*, where* $I_1(u, v)$ *is the first fundamental form of* S_1 *and* $I_2(u, v)$ *is the first fundamental form of* S_2 *induced by* I_1*. We say that* S_1 *and* S_2 *are* **isometric** *if there exists an isometry between* S_1 *and* S_2*.*

Theorem 3.5.2 (Theorema egregium). *Let S_1 and S_2 be two regular surfaces. Suppose S_1 and S_2 are isometric, that is, there exists isometry $f : S_1 \to S_2$ between S_1 and S_2. Then for any $p \in S_1$, the Gaussian curvature of S_1 at p is equal to the Gaussian curvature of S_2 at $f(p)$. In other words,*

$$K(f(p)) = K(p)$$

for any $p \in S_1$.



Isometry

bijective

$I_1 = I_2$

K was defined using

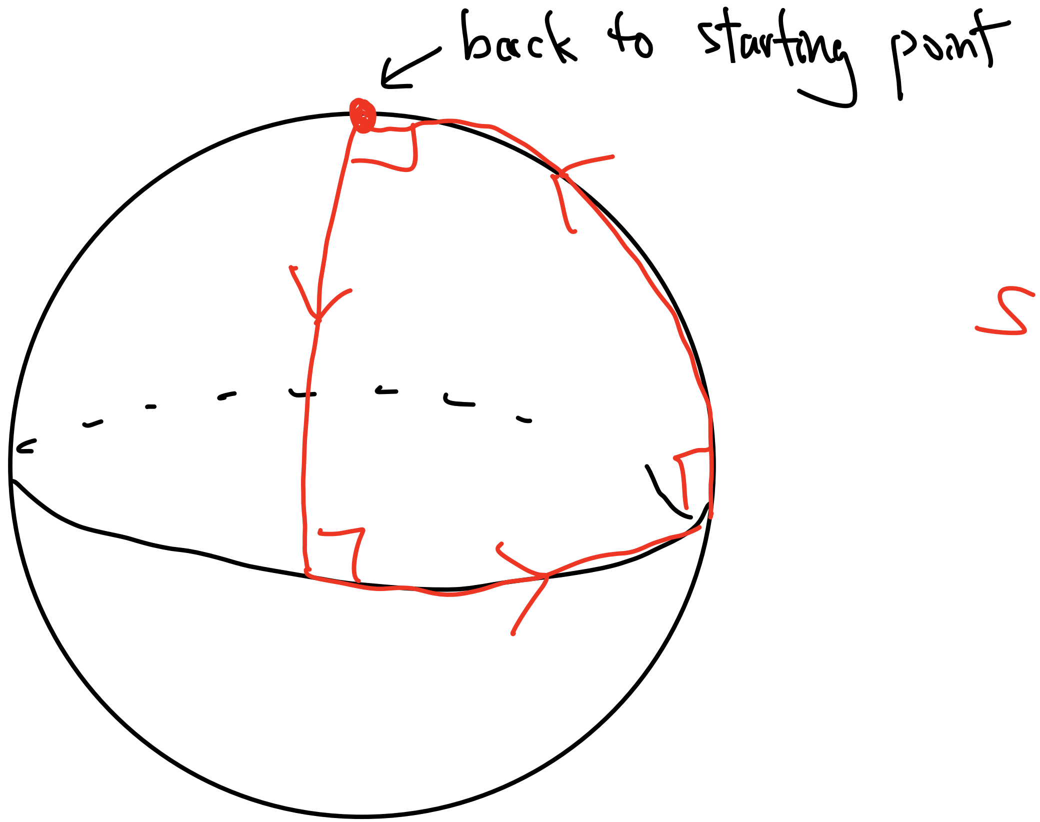
$\vec{n}$, I, II : $K = \dfrac{\det \mathrm{II}}{\det \mathrm{I}}$

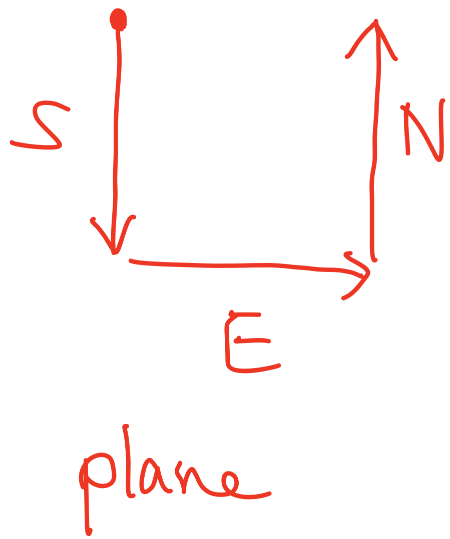
Thm 3.5.2 $\Longrightarrow$ K is indep't of $\vec{n}$, II

Pf

$$K = \frac{1}{4(EG - F^2)^2}\left(\begin{vmatrix} -E_{vv} + 2F_{uv} - G_{uu} & E_u & 2F_u - E_v \\ 2F_v - G_u & E & F \\ G_v & F & G \end{vmatrix} - \begin{vmatrix} 0 & E_v & G_u \\ E_v & E & F \\ G_u & F & G \end{vmatrix} \right)$$

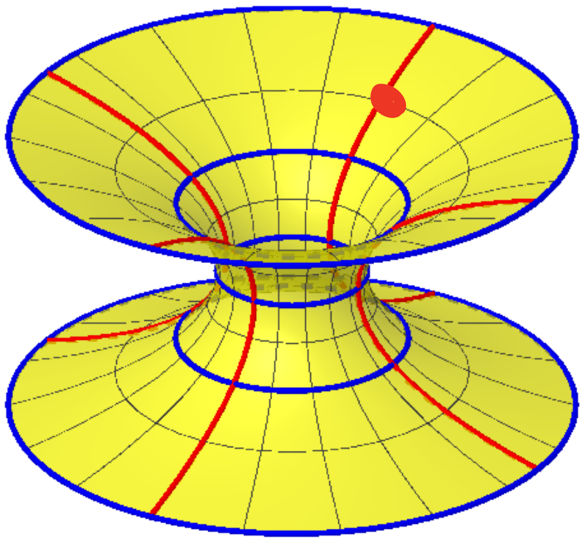
Thm 3.5.2 $\Rightarrow$ Measuring distance can be used to compute curvature





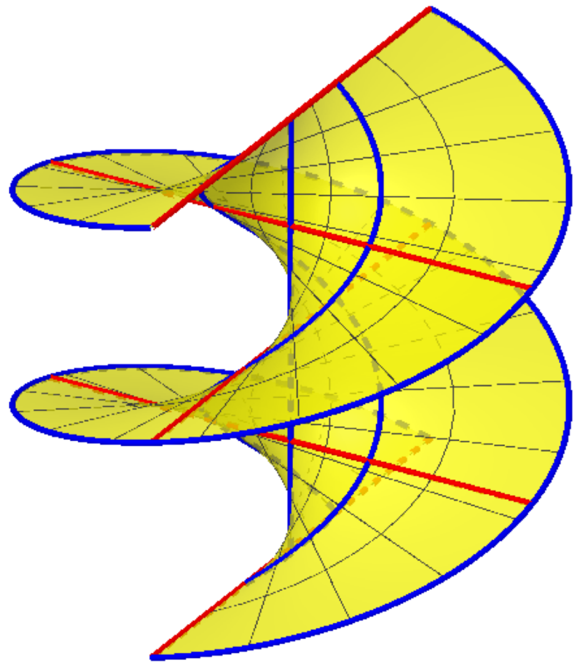
Example 3.5.3 (Isometry between catenoid and helicoid)

Catenoid



$\xrightarrow{f}$

Helicoid



$\mathbf{x}_1(\theta, v) = (\cosh v \cos\theta, \cosh v \sin\theta, v)$

$\mathbf{x}_2(\theta, v) = (\sinh v \cos\theta, \sinh v \sin\theta, \theta)$

$(\theta, v) \in \underbrace{(0, 2\pi) \times \mathbb{R}}_{O}$

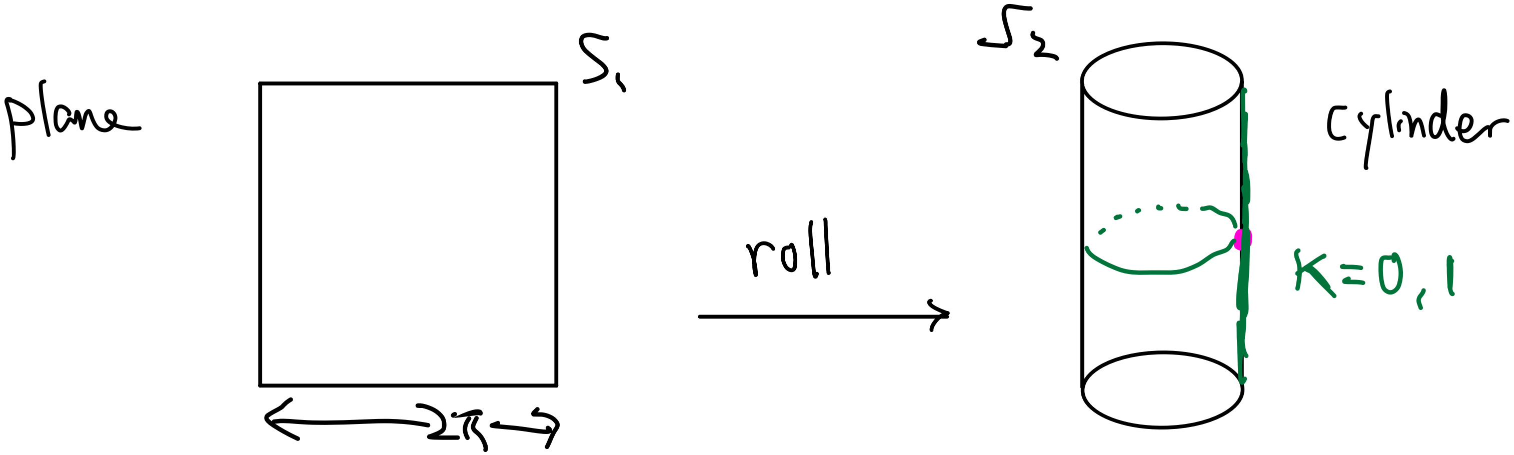
$f(x_1(\theta,v)) = x_2(\theta,v)$

$I_1(\theta,v) = I_2(\theta,v) = \begin{pmatrix} \cosh^2 v & 0 \\ 0 & \cosh^2 v \end{pmatrix} \Rightarrow$ Isometry $\Rightarrow$ K same

$I_1(\theta, v) = I_2(\theta, v) = \begin{pmatrix} \cosh^2 v & 0 \\ 0 & \cosh^2 v \end{pmatrix}$

$K = -\dfrac{1}{\cosh^4 v}.$

Catenoid and Helicoid are both minimal surface and thus have mean curvature identically zero. However, the mean curvature of two isometric surfaces may not be identical. For example, a cylindrical surface and a plane are isometric but a cylindrical surface has nonzero mean curvature while that of a plane is zero.



$(\theta, z) \in (0, 2\pi) \times \mathbb{R}$

$x_1(\theta, z) = (\theta, 0, z)$ $\qquad$ $x_2(\theta, z) = (\cos\theta, \sin\theta, z)$

$$I_1 = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow f \text{ is isometry} \Longrightarrow \text{same } K(\theta, z)$$

<u>Rmk</u> Different mean curvature $H_1 = 0$ $\qquad$ $H_2 = \frac{1}{2}$ (or $-\frac{1}{2}$)

Theorem 3.5.4. *Let $\mathbf{x}(u,v)$ be a regular parametrized surface. Then*

$$K = \frac{1}{4(EG-F^2)^2}\left(\begin{vmatrix} -E_{vv}+2F_{uv}-G_{uu} & E_u & 2F_u - E_v \\ 2F_v - G_u & E & F \\ G_v & F & G \end{vmatrix} - \begin{vmatrix} 0 & E_v & G_u \\ E_v & E & F \\ G_u & F & G \end{vmatrix} \right).$$

In particular, if $F = 0$ is identically zero, then

$$K = -\frac{1}{2\sqrt{EG}}\left[\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right].$$

Proof. Since $\det(I) = EG - F^2 = \|\mathbf{x}_u \times \mathbf{x}_v\|^2$ and $\|\mathbf{x}_u \times \mathbf{x}_v\|\mathbf{n} = \mathbf{x}_u \times \mathbf{x}_v$,

$$\begin{aligned}
& K(EG-F^2)^2 \\
= \ & \det(I)\det(II) \\
= \ & \|\mathbf{x}_u \times \mathbf{x}_v\|^2 \begin{vmatrix} \langle \mathbf{x}_{uu}, \mathbf{n}\rangle & \langle \mathbf{x}_{uv}, \mathbf{n}\rangle \\ \langle \mathbf{x}_{vu}, \mathbf{n}\rangle & \langle \mathbf{x}_{vv}, \mathbf{n}\rangle \end{vmatrix} \\
= \ & \begin{vmatrix} \langle \mathbf{x}_{uu}, \|\mathbf{x}_u \times \mathbf{x}_v\|\mathbf{n}\rangle & \langle \mathbf{x}_{uv}, \|\mathbf{x}_u \times \mathbf{x}_v\|\mathbf{n}\rangle \\ \langle \mathbf{x}_{vu}, \|\mathbf{x}_u \times \mathbf{x}_v\|\mathbf{n}\rangle & \langle \mathbf{x}_{vv}, \|\mathbf{x}_u \times \mathbf{x}_v\|\mathbf{n}\rangle \end{vmatrix} \\
= \ & \begin{vmatrix} \langle \mathbf{x}_{uu}, \mathbf{x}_u \times \mathbf{x}_v\rangle & \langle \mathbf{x}_{uv}, \mathbf{x}_u \times \mathbf{x}_v\rangle \\ \langle \mathbf{x}_{vu}, \mathbf{x}_u \times \mathbf{x}_v\rangle & \langle \mathbf{x}_{vv}, \mathbf{x}_u \times \mathbf{x}_v\rangle \end{vmatrix} \\
= \ & \begin{vmatrix} \langle \mathbf{x}_{uu}, \mathbf{x}_{vv}\rangle - \langle \mathbf{x}_{uv}, \mathbf{x}_{uv}\rangle & \langle \mathbf{x}_{uu}, \mathbf{x}_u\rangle & \langle \mathbf{x}_{uu}, \mathbf{x}_v\rangle \\ \langle \mathbf{x}_{vv}, \mathbf{x}_u\rangle & \langle \mathbf{x}_u, \mathbf{x}_u\rangle & \langle \mathbf{x}_u, \mathbf{x}_v\rangle \\ \langle \mathbf{x}_{vv}, \mathbf{x}_v\rangle & \langle \mathbf{x}_v, \mathbf{x}_u\rangle & \langle \mathbf{x}_v, \mathbf{x}_v\rangle \end{vmatrix} \\
& - \begin{vmatrix} 0 & \langle \mathbf{x}_{uv}, \mathbf{x}_u\rangle & \langle \mathbf{x}_{uv}, \mathbf{x}_v\rangle \\ \langle \mathbf{x}_{vu}, \mathbf{x}_u\rangle & \langle \mathbf{x}_u, \mathbf{x}_u\rangle & \langle \mathbf{x}_u, \mathbf{x}_v\rangle \\ \langle \mathbf{x}_{vu}, \mathbf{x}_v\rangle & \langle \mathbf{x}_v, \mathbf{x}_u\rangle & \langle \mathbf{x}_v, \mathbf{x}_v\rangle \end{vmatrix}. \ (\text{Proposition } 1.3.17)
\end{aligned}$$

Observe that by product rule (Proposition 1.3.34),

$$\begin{aligned}
\begin{pmatrix} E_u & F_u \\ F_u & G_u \end{pmatrix} &= \frac{\partial}{\partial u}\begin{pmatrix} E & F \\ F & G \end{pmatrix} \\
&= \frac{\partial}{\partial u}\begin{pmatrix} \langle \mathbf{x}_u, \mathbf{x}_u\rangle & \langle \mathbf{x}_u, \mathbf{x}_v\rangle \\ \langle \mathbf{x}_v, \mathbf{x}_u\rangle & \langle \mathbf{x}_v, \mathbf{x}_v\rangle \end{pmatrix} \\
&= \begin{pmatrix} \langle \mathbf{x}_{uu}, \mathbf{x}_u\rangle & \langle \mathbf{x}_{uu}, \mathbf{x}_v\rangle \\ \langle \mathbf{x}_{vu}, \mathbf{x}_u\rangle & \langle \mathbf{x}_{vu}, \mathbf{x}_v\rangle \end{pmatrix} + \begin{pmatrix} \langle \mathbf{x}_u, \mathbf{x}_{uu}\rangle & \langle \mathbf{x}_u, \mathbf{x}_{vu}\rangle \\ \langle \mathbf{x}_v, \mathbf{x}_{uu}\rangle & \langle \mathbf{x}_v, \mathbf{x}_{vu}\rangle \end{pmatrix} \\
&= \begin{pmatrix} 2\langle \mathbf{x}_{uu}, \mathbf{x}_u\rangle & \langle \mathbf{x}_{uu}, \mathbf{x}_v\rangle + \langle \mathbf{x}_{uv}, \mathbf{x}_u\rangle \\ \langle \mathbf{x}_{uu}, \mathbf{x}_v\rangle + \langle \mathbf{x}_{uv}, \mathbf{x}_u\rangle & 2\langle \mathbf{x}_{vu}, \mathbf{x}_v\rangle \end{pmatrix}.
\end{aligned}$$

Similarly

$$\begin{pmatrix} E_v & F_v \\ F_v & G_v \end{pmatrix} = \begin{pmatrix} 2\langle \mathbf{x}_{uv}, \mathbf{x}_u\rangle & \langle \mathbf{x}_{vv}, \mathbf{x}_u\rangle + \langle \mathbf{x}_{uv}, \mathbf{x}_v\rangle \\ \langle \mathbf{x}_{vv}, \mathbf{x}_u\rangle + \langle \mathbf{x}_{uv}, \mathbf{x}_v\rangle & 2\langle \mathbf{x}_{vv}, \mathbf{x}_v\rangle \end{pmatrix}.$$

Combining the above two equalities, we obtain

$$\begin{cases}
\langle \mathbf{x}_{uu}, \mathbf{x}_u\rangle = \dfrac{E_u}{2}, \\
\langle \mathbf{x}_{uv}, \mathbf{x}_u\rangle = \dfrac{E_v}{2}, \\
\langle \mathbf{x}_{uv}, \mathbf{x}_v\rangle = \dfrac{G_u}{2}, \\
\langle \mathbf{x}_{vv}, \mathbf{x}_v\rangle = \dfrac{G_v}{2}, \\
\langle \mathbf{x}_{uu}, \mathbf{x}_v\rangle = F_u - \langle \mathbf{x}_{uv}, \mathbf{x}_u\rangle = F_u - \dfrac{E_v}{2}, \\
\langle \mathbf{x}_{vv}, \mathbf{x}_u\rangle = F_v - \langle \mathbf{x}_{uv}, \mathbf{x}_v\rangle = F_v - \dfrac{G_u}{2}.
\end{cases}$$

Moreover by considering the second derivative of $F = \langle \mathbf{x}_u, \mathbf{x}_v\rangle$ with respect to u, v, we have

$$\begin{aligned}
\langle \mathbf{x}_{uu}, \mathbf{x}_{vv}\rangle &= \frac{\partial}{\partial u}\langle \mathbf{x}_{vv}, \mathbf{x}_u\rangle - \langle \mathbf{x}_{vvu}, \mathbf{x}_u\rangle \\
&= \frac{\partial}{\partial u}\left(F_v - \frac{G_u}{2}\right) - \left(\frac{\partial}{\partial v}\langle \mathbf{x}_{uv}, \mathbf{x}_u\rangle - \langle \mathbf{x}_{uv}, \mathbf{x}_{uv}\rangle\right) \\
&= F_{uv} - \frac{G_{uu}}{2} - \left(\frac{\partial}{\partial v}\frac{E_v}{2} - \langle \mathbf{x}_{uv}, \mathbf{x}_{uv}\rangle\right) \\
&= -\frac{E_{vv}}{2} + F_{uv} - \frac{G_{uu}}{2} + \langle \mathbf{x}_{uv}, \mathbf{x}_{uv}\rangle
\end{aligned}$$

which implies

$$\langle \mathbf{x}_{uu}, \mathbf{x}_{vv}\rangle - \langle \mathbf{x}_{uv}, \mathbf{x}_{uv}\rangle = -\frac{E_{vv}}{2} + F_{uv} - \frac{G_{uu}}{2}.$$

Therefore

$$K(EG-F^2)^2 = \left(\begin{vmatrix} -\frac{E_{vv}}{2} + F_{uv} - \frac{G_{uu}}{2} & \frac{E_u}{2} & F_u - \frac{E_v}{2} \\ F_v - \frac{G_u}{2} & E & F \\ \frac{G_v}{2} & F & G \end{vmatrix} - \begin{vmatrix} 0 & \frac{E_v}{2} & \frac{G_u}{2} \\ \frac{E_v}{2} & E & F \\ \frac{G_u}{2} & F & G \end{vmatrix}\right)$$

as desire. If particular, if $F = 0$, then

$$\begin{aligned}
K &= \frac{1}{4E^2G^2}\left(\begin{vmatrix} -E_{vv} - G_{uu} & E_u & -E_v \\ -G_u & E & 0 \\ G_v & 0 & G \end{vmatrix} - \begin{vmatrix} 0 & E_v & G_u \\ E_v & E & 0 \\ G_u & 0 & G \end{vmatrix}\right) \\
&= \frac{1}{4E^2G^2}\left(-EGE_{vv} - EGG_{uu} + GE_uG_u + EE_vG_v + GE_v^2 + EG_u^2\right) \\
&= -\frac{E_{vv}}{4EG} - \frac{G_{uu}}{4EG} + \frac{E_uG_u}{4E^2G} + \frac{E_vG_v}{4EG^2} + \frac{E_v^2}{4E^2G} + \frac{G_u^2}{4EG^2}.
\end{aligned}$$

Observe that

$$\begin{cases}
\left(\dfrac{E_v}{\sqrt{EG}}\right)_v = \dfrac{E_{vv}}{\sqrt{EG}} - \dfrac{E_v^2}{2E\sqrt{EG}} - \dfrac{E_vG_v}{2G\sqrt{EG}} \\
\left(\dfrac{G_u}{\sqrt{EG}}\right)_u = \dfrac{G_{uu}}{\sqrt{EG}} - \dfrac{G_u^2}{2G\sqrt{EG}} - \dfrac{E_uG_u}{2E\sqrt{EG}}.
\end{cases}$$

Hence

$$\begin{aligned}
&\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u \\
&= \frac{E_{vv}}{\sqrt{EG}} - \frac{E_v^2}{2E\sqrt{EG}} - \frac{E_vG_v}{2G\sqrt{EG}} + \frac{G_{uu}}{\sqrt{EG}} - \frac{G_u^2}{2G\sqrt{EG}} - \frac{E_uG_u}{2E\sqrt{EG}} \\
&= -2\sqrt{EG}\left(-\frac{E_{vv}}{4EG} + \frac{E_v^2}{4E^2G} + \frac{E_vG_v}{4EG^2} - \frac{G_{uu}}{4EG} + \frac{G_u^2}{4EG^2} + \frac{E_uG_u}{4E^2G}\right) \\
&= -2K\sqrt{EG}
\end{aligned}$$

and the result follows. $\square$

3.6 Gauss-Bonnet theorem

$\int_a^b f'(t)\,dt = f(b) - f(a)$

Theorem 3.6.6 (Gauss-Bonnet theorem). *Let S be a simple closed regular surface in $\mathbb{R}^3$. Then*

$$\iint_S K\,dA = 2\pi \underbrace{\chi(S)}$$

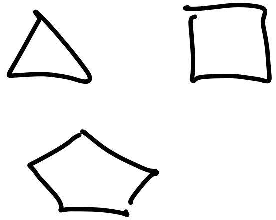
local: curvature   Euler characteristics (Global: shape)

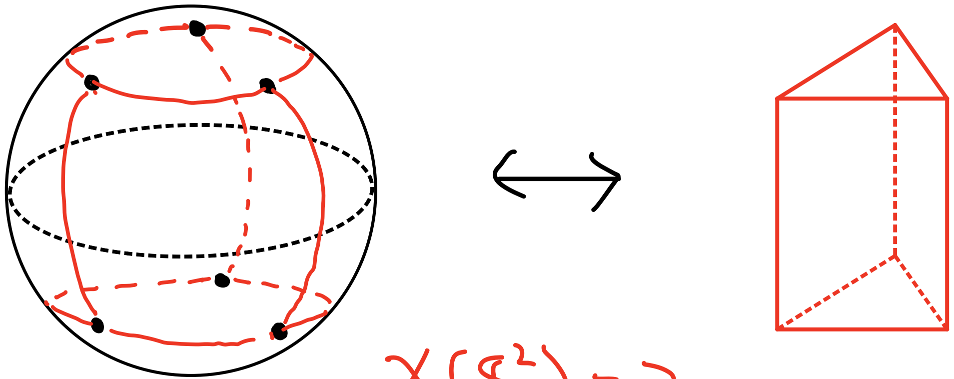
Definition 3.6.1 (Euler characteristic). *The* **Euler characteristic** *of a closed surface S is*

$$\chi(S) = v - e + f$$

where v, e and f are the number of vertices, edges and faces of a polyhedron modeled on S.

polygon



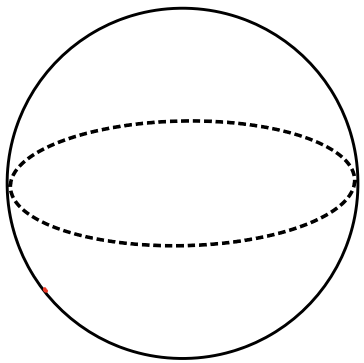


$v = 6$

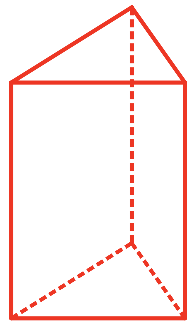
$e = 9 \quad f = 5$

$v - e + f = 2$

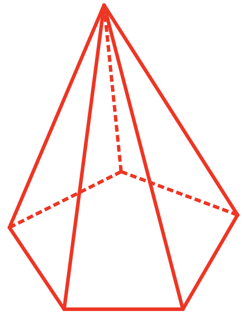
$\chi(S^2) = 2$



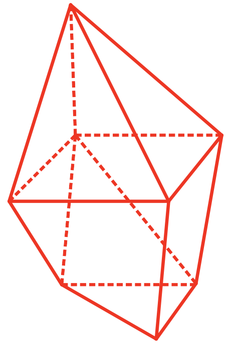
$\longleftrightarrow$



$v=6$
$e=9$
$f=5$

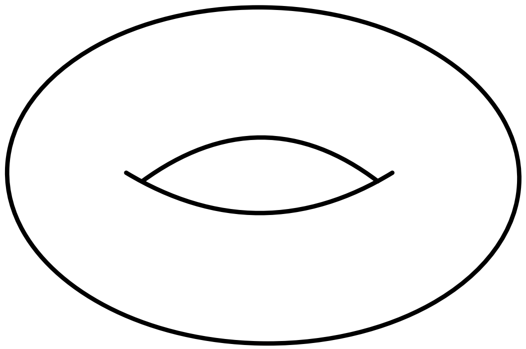


$v=6$
$e=10$
$f=6$



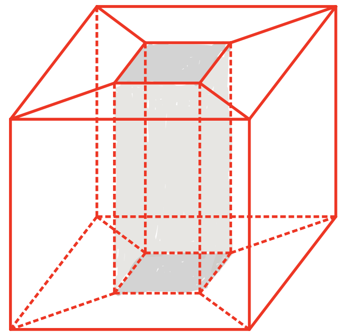
$v=8$
$e=16$
$f=10$

$$\chi(S^2) = v - e + f$$
$$= 2$$



$\longleftrightarrow$

12
8
12



$v=16$
$e=32$
$f=16$

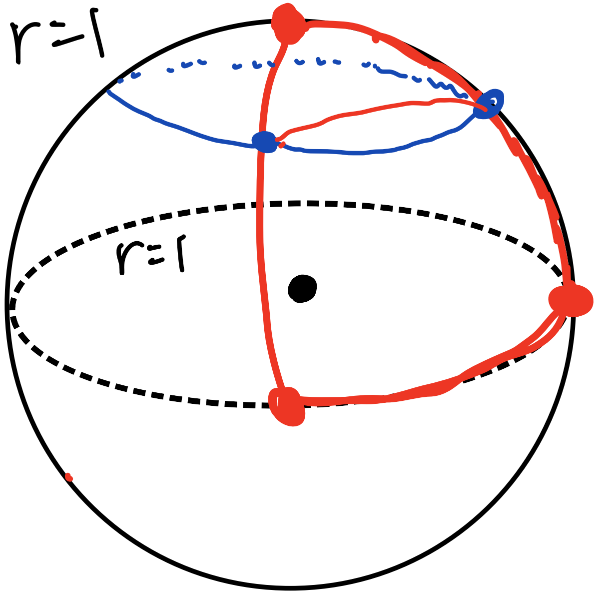
$$\chi(T) = v - e + f = 0$$

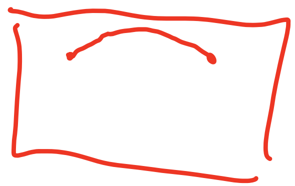
Theorem 3.6.2 (Area of polygon on unit sphere). *Let α, β, γ be the interior angles of a triangle, with edges being great circular arcs*[12]*, on the unit sphere and A be the area of the triangle. Then*

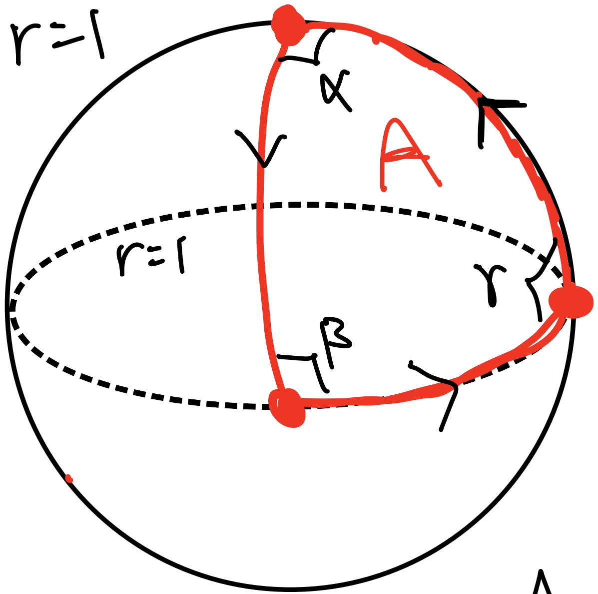
$$\alpha + \beta + \gamma = A + \pi.$$

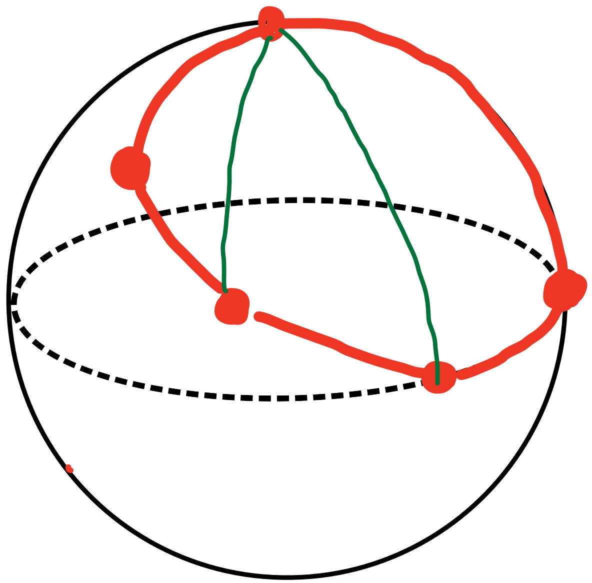
More generally, Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the interior angles of a polygon with n edges, which are great circular arcs, on the unit sphere and A be the area of the polygon. Then

$$\alpha_1 + \alpha_2 + \cdots + \alpha_n = A + (n-2)\pi.$$









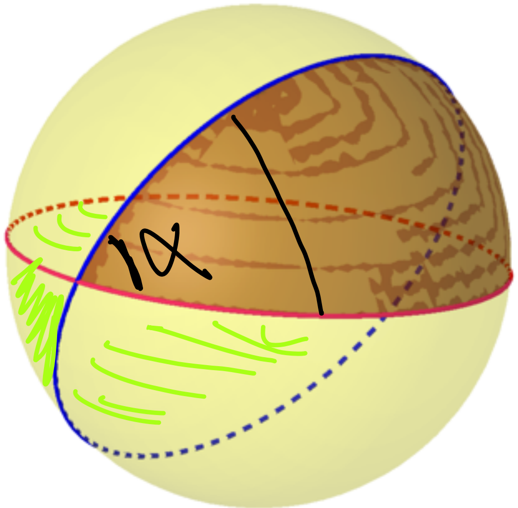
$$\alpha+\beta+\gamma = \frac{3\pi}{2} = \overset{A}{\overset{\|}{\frac{4\pi}{8}}} + \pi$$

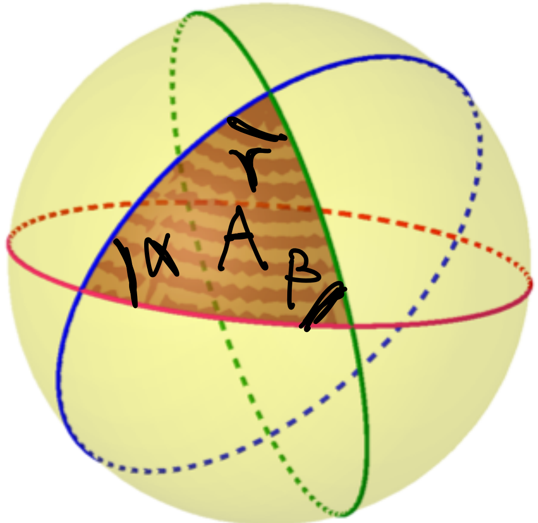
$n=5.$

Subdivide into $(n-2)=3$ triangles

Theorem 3.6.2 (Area of polygon on unit sphere). *Let α, β, γ be the interior angles of a triangle, with edges being great circular arcs*[12]*, on the unit sphere and A be the area of the triangle. Then*

$$\alpha + \beta + \gamma = A + \pi.$$





Area of brown region

$= 4\pi \cdot \frac{\alpha}{2\pi} = 2\alpha$

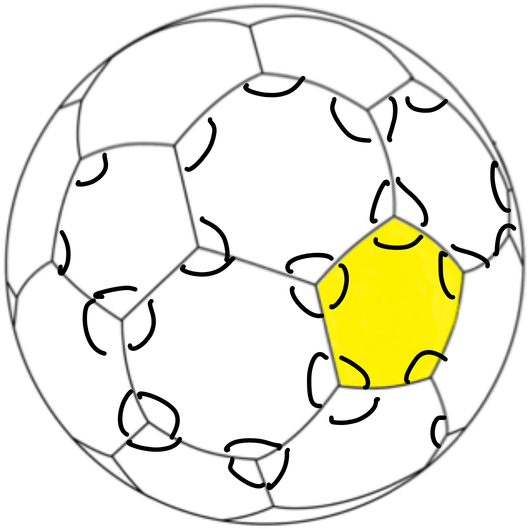
$2(2\alpha) + 2(2\beta) + 2(2\gamma) = 4\pi$

$+ 2A + 2A$

$\alpha + \beta + \gamma = \pi + A$

Theorem 3.6.3 (Euler characteristic of sphere). *A polyhedron which is modeled on a sphere has Euler characteristic* $\chi = 2$.

Proof. Consider a polyhedron modeled on the unit sphere. By deforming the edges, we may assume that the edges are great circular arcs on the unit sphere. Let v, e and f be the number of vertices, edges and faces of the polyhedron. Suppose the k-th face, $k = 1, 2, \ldots, f$, is a polygon with e_k edges, e_k interior angles $\alpha_{k_1}, \alpha_{k_2}, \ldots, \alpha_{k_{e_k}}$ and has area equal to A_k. By Theorem 3.6.2, we have



$$\sum_{i=1}^{e_k} \alpha_{k_i} = (e_k - 2)\pi + A_k$$

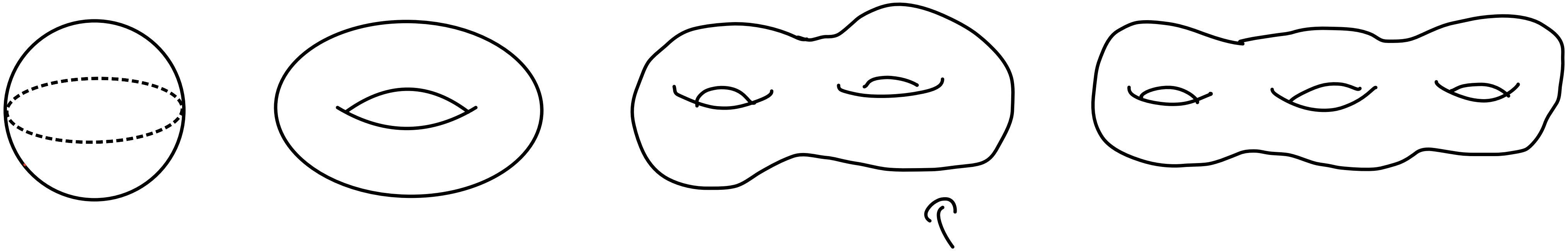
$$\underbrace{\sum_{k=1}^{f}\sum_{i=1}^{e_k} \alpha_{k_i}}_{2\pi v} = \underbrace{\sum_{k=1}^{f} e_k\pi}_{2\pi e} - \underbrace{2\sum_{k=1}^{f} \pi}_{2\pi f} + \underset{+\,4\pi}{\sum_{k=1}^{f} A_k}.$$

k-th face, e_k edges

1, 2, 3, ..., f faces

$$v - e + f = 2$$

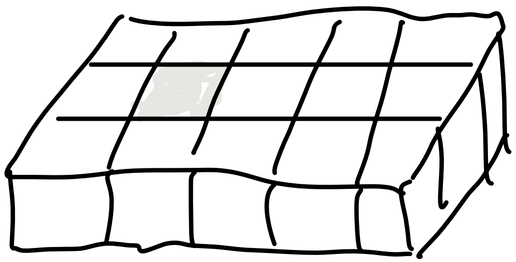
Genus of closed surfaces (Number of holes)



$g=0$	$g=1$	$g=2$	$g=3$
$\chi=2$	$\chi=0$	$\chi=-2$	$\chi=-4$

Theorem 3.6.4 (Euler characteristic of simple closed surface)**.** *Let S be a simple closed surface of genus g. Then the Euler characteristic of S is*

$$\chi(S) = 2 - 2g.$$



$\chi = v - e + f$

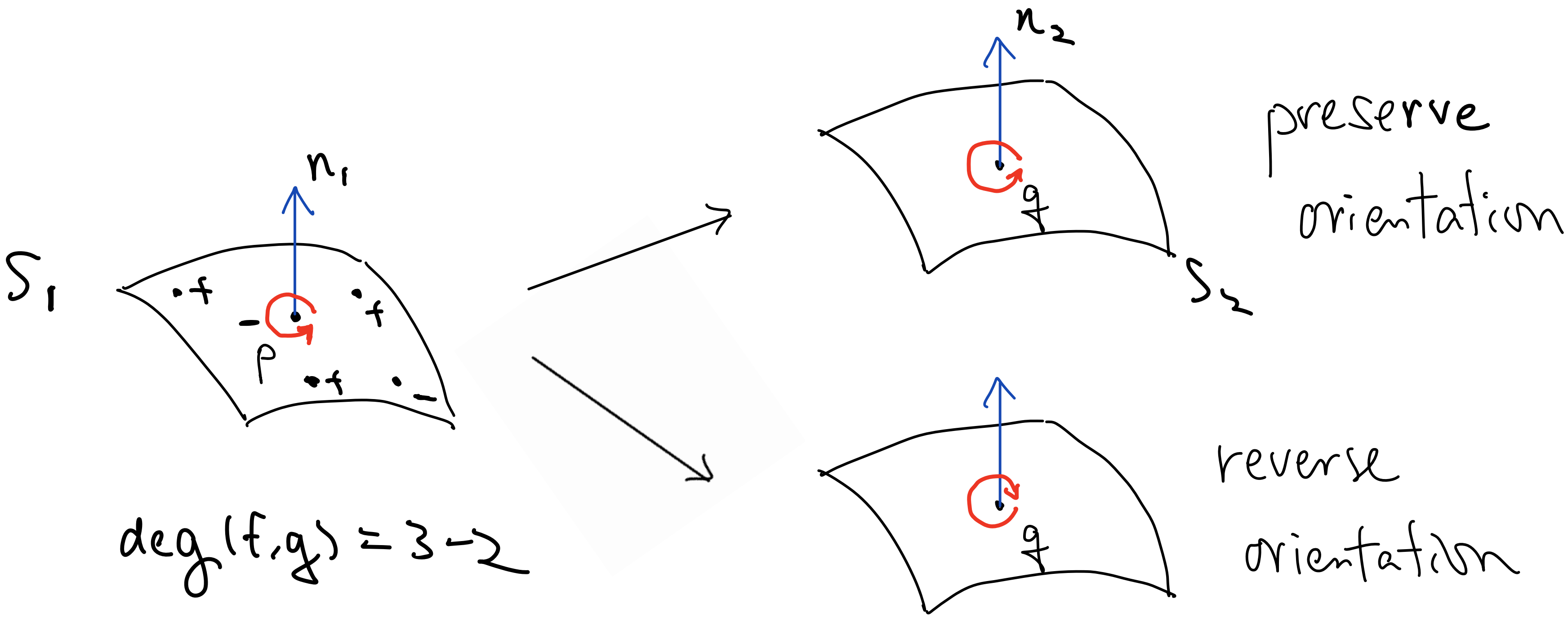
$\downarrow$ +4 $\downarrow$ +2   add a hole $\Rightarrow$ χ -2

Degree of a map between surfaces

Let S_1 and S_2 be two simple closed surface in $\mathbb{R}^3$. Let $f : S_1 \to S_2$ be a continuous map from S_1 to S_2. For $q \in S_2$, we define the degree of f at q to be the integer

$$\deg(f,q) = \begin{array}{l} \text{number of preimages of } q \text{ preserving orientation} \\ -\text{number of preimages of } q \text{ reversing orientation} \end{array}.$$

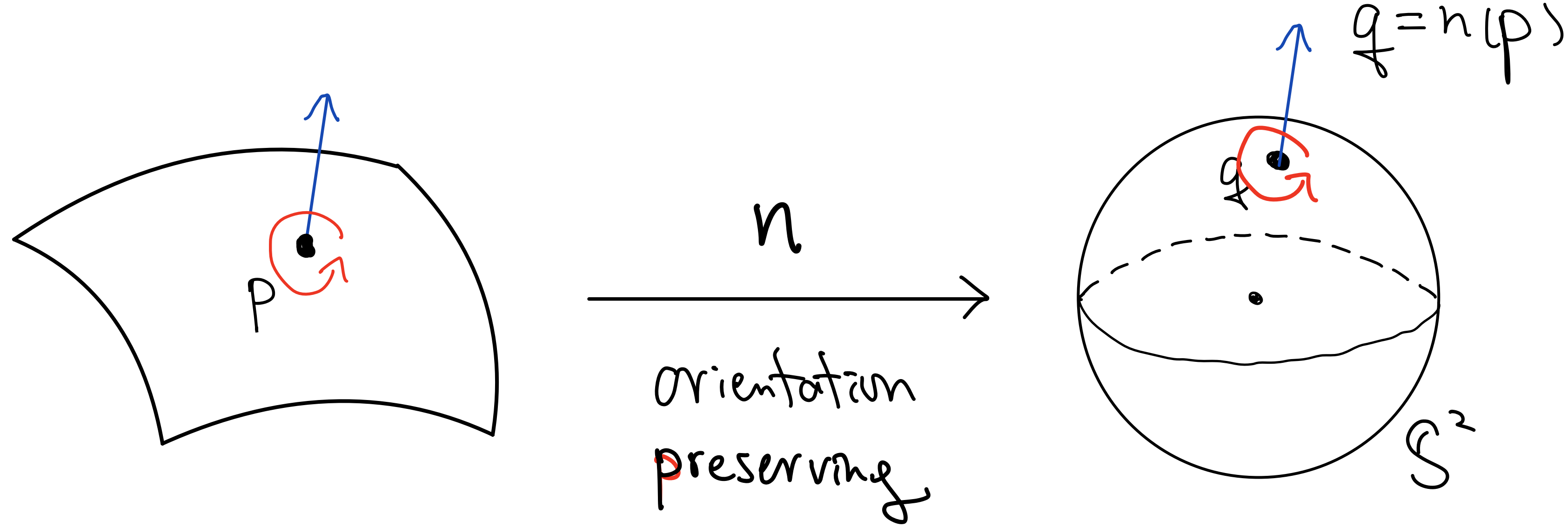
Same for any q with finite preimage. Called $\deg(f)$



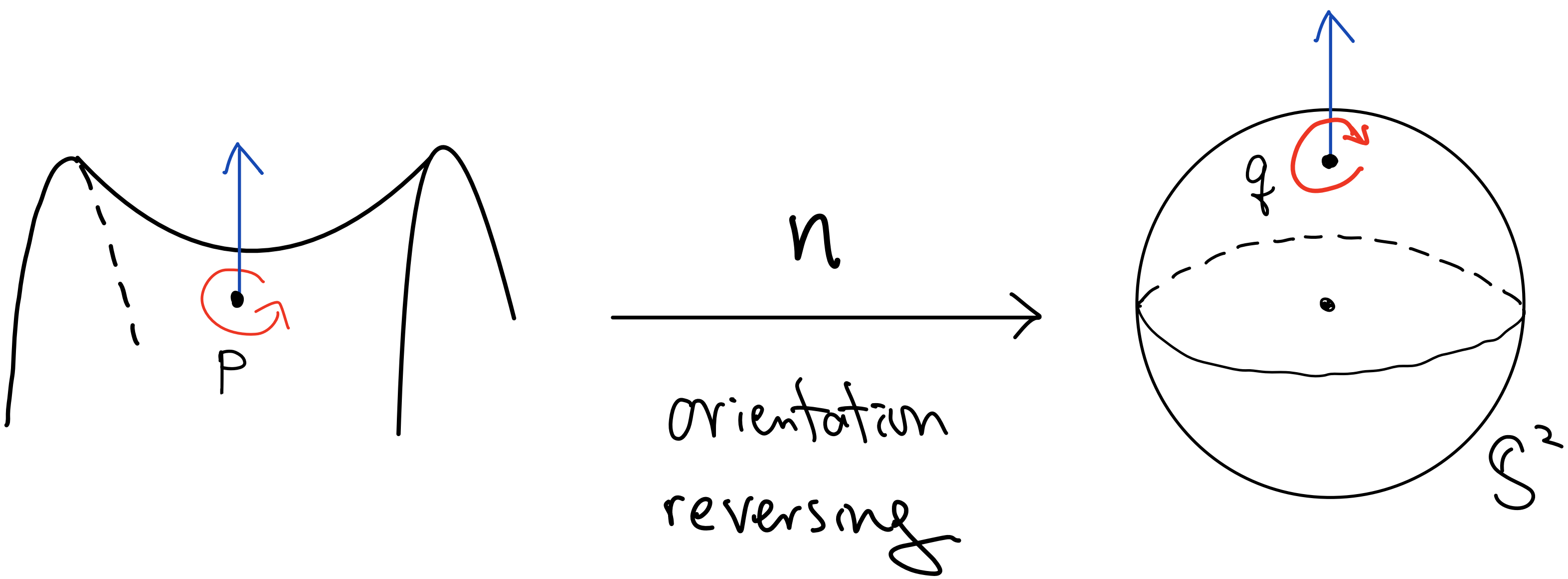
$\deg(f,q) = 3-2$

Degree of Gauss Map

$K(p) > 0$

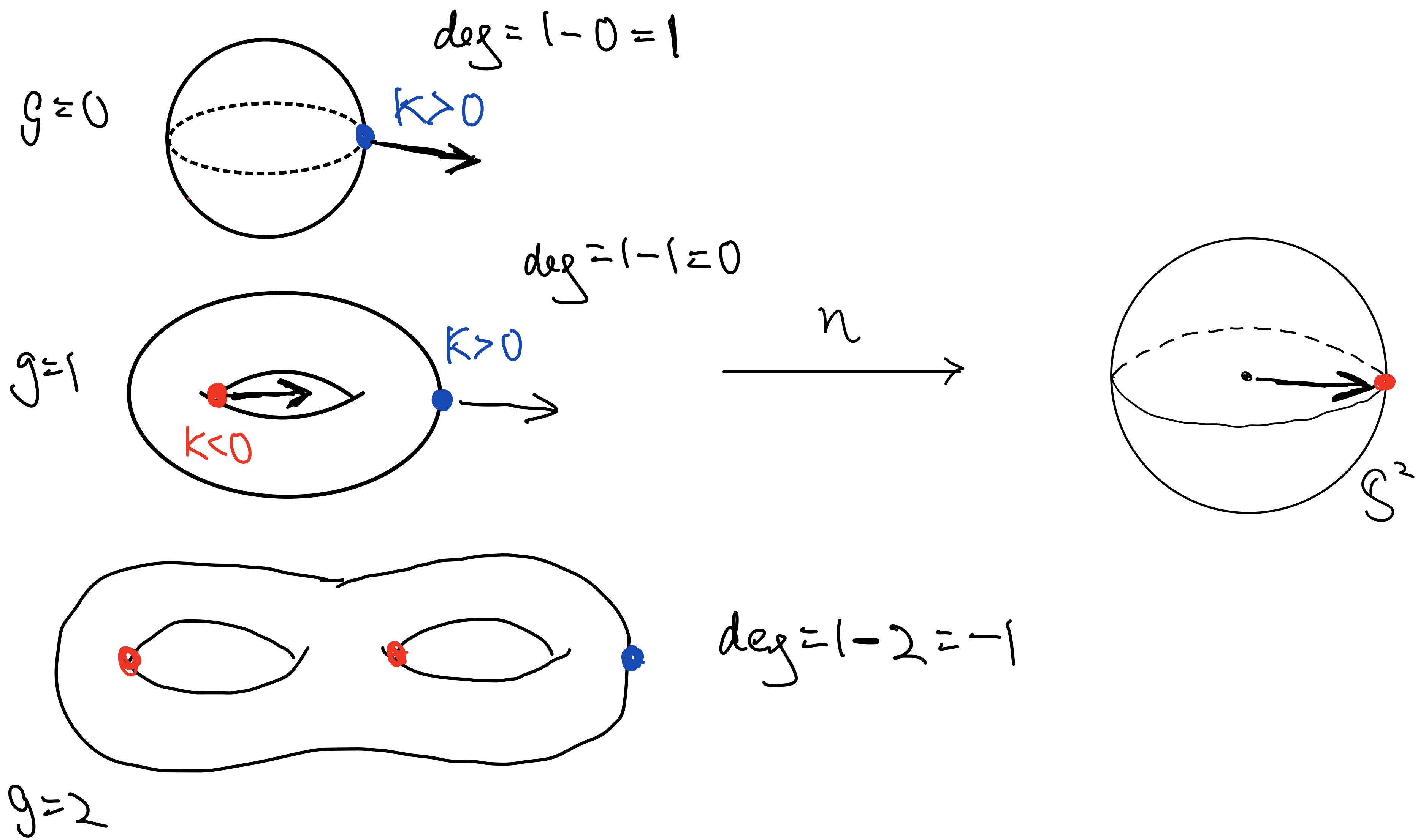


$K(p) < 0$



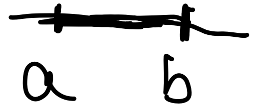
Theorem 3.6.5 (Degree of Gauss map of simple closed regular surface). *Let S be a simple closed surface of genus g. The the degree of Gauss map of S is*

$$\deg(\mathbf{n}) = 1 - g.$$



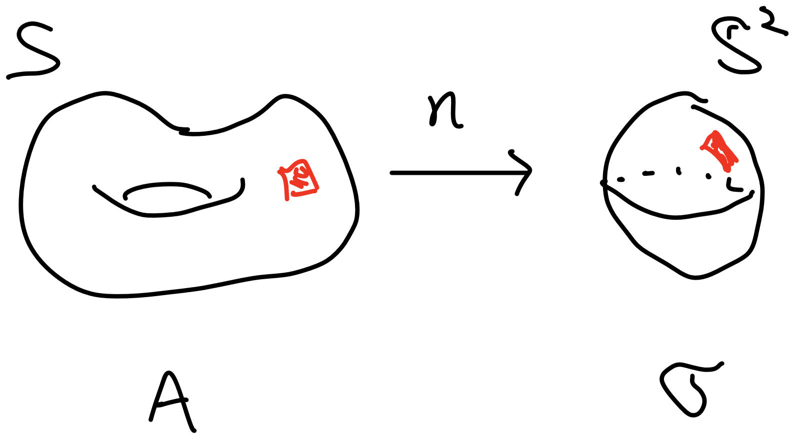
Theorem 3.6.6 (Gauss-Bonnet theorem). *Let S be a simple closed regular surface in $\mathbb{R}^3$. Then*

$$\iint_S K dA = 2\pi\chi(S)$$



where K is the Gaussian curvature, $\chi(S)$ is the Euler characteristic of S and $dA = \sqrt{\det(I)}dudv$ is the surface area element. In particular, if S is homeomorphic[13] *to the sphere S^2, then $\chi(S) = 2$ and*

$$\iint_S K dA = 4\pi.$$



Pf

$$\iint_S K dA = \iint_S \frac{d\sigma}{dA} dA$$

$$= \iint_S d\sigma$$

$$= \deg(n) \iint_{S^2} d\sigma$$

$$= (1-g)(4\pi) = 2\pi(2-2g) = 2\pi\chi(S)$$